Homological Algebra, its relationship with distributive lattices and relations

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Abstract

We want to explore aspects of coherence in homological algebra, that already appear in the classical situation of abelian groups or abelian categories. Lattices of subobjects play an important role in the study of homological systems, including all the structures that give rise to spectral sequences. A parallel role is played by semigroups of endorelations.

These links rest on the fact that many such systems, but not all of them, live in *distributive sublattices* of the modular lattices of subobjects of the system. The property of distributivity allows one to work with induced morphisms in an automatically consistent way, as we prove in a 'Coherence Theorem for homological algebra' [G1, G2]. The same property of distributivity also permits representations of homological structures by means of sets and *lattices of subsets*; this yields a precise foundation for the heuristic tool of Zeeman diagrams [Ze], as universal models of spectral sequences.

This approach, based on lattices of subobjects, has from the beginning a sort of 'projective character', since a projectivity between groups is an isomorphism between their lattices of subgroups.

All this can be *applied* to abelian categories, but cannot be developed *inside* this setting: we need a more general framework, that has often been viewed as an intermediate step not important in itself: exact categories in the sense of Puppe and Mitchell [Mi], or *p-exact categories*. It is a selfdual theory, based on kernels and cokernels, where the existence of cartesian products is not assumed. The lattices of subobjects are always modular, but there are important cases where all of them are distributive - a fact that cannot happen in a non-trivial abelian category.

This extension attains various non-abelian categories, like the category of projective spaces on a fixed field (again, a 'projective' aspect of our extension). More importantly, on a structural ground, one can here establish and analyse

the ‘subterranean’ aspects of coherence mentioned above. In fact, one of the main tools of our approach is the functor of subobjects, that - even for abelian groups - takes values in a category of modular lattices and suitable adjunctions, *that is p-exact* and can*not* even be exactly embedded in an abelian category. By a derived construction, the *distributive expansion* of a p-exact category, we can apply the coherence theorem to any p-exact category, including the abelian ones. Finally, most of the universal models that we construct live in the category of sets and partial bijections, a sort of universal distributive p-exact category.

Altogether, this approach is essentially addressed to study the structure of the homological algebra of abelian categories, in a non-abelian extension where this structure can be better analysed. This matter has been extended in [G3] to ‘strongly’ non-abelian situations, where it becomes possible to investigate the spectral sequences of unstable homotopy.

It should be noticed that these settings are quite distinct from the more usual *affine* generalisations of the abelian framework,based on finite limits: Barr-exact categories, their extensions and variations. There seems to be a sort of opposition between a projective and an affine approach.

References

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